JOURNAL OF SPACECRAFT AND ROCKETS
Vol. 36, No. 4, July–August 1999

Low-Thrust Trajectory Optimization Based on Epoch Eccentric Longitude Formulation

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The mathematics of trajectory optimization based on the use of nonsingular orbit elements involving the eccentric longitude at epoch as the sixth element are fully derived. The epoch eccentric longitude and epoch mean longitude are fundamental nonsingular orbit elements, which stay constant in the absence of perturbations. These formulations constitute the basis from which the current time formulations are derived and, therefore, are important from a theoretical point of view. They are also beneficial in minimum-fuel problems during the natural coasting parts of the trajectory, where the adjoint equations need not be propagated by either analytical or numerical integration. The state and adjoint differential equations are explicit functions of time in this formulation that involves natural orbit elements with the optimal Hamiltonian, also varying in time. The mathematics of this epoch formulation provides added insight into the problem of trajectory optimization by relating the various assumptions used in generating the differential equations for the adjoint variables that correspond to various sets of orbital elements. Furthermore, the function that defines the transversality condition at the end time in minimum-time problems is shown to remain constant during the optimal transfer providing a further check in accepting a converged trajectory as truly optimal. This formulation is also related to the one that uses the current eccentric longitude as the sixth state variable, and the mathematical relationship between the Hamiltonian and the Lagrange multipliers of these two formulations is also shown. A pair of continuous constant acceleration minimum-time transfer examples are duplicated using this new formulation to validate the mathematical derivations.

Nomenclature

F	= eccentric longitude, $E + \omega + \Omega$, rad
f	= thrust magnitude, N
$\hat{f}, \hat{g}, \hat{w}$	= unit vectors defining equinoctial frame
f_t	= thrust acceleration, km/s^2
G	= $(1 - h^2 - k^2)^{1/2}$
h, k	= $e \sin(\omega + \Omega)$, $e \cos(\omega + \Omega)$, respectively
K	= $1 + p^2 + q^2$
L	= true longitude, $\theta^* + \omega + \Omega$, rad
m	= spacecraft mass, kg
n	= orbit mean motion, $\mu^{1/2}a^{-3/2}$, rad/s
p, q	= $\tan(i/2)\sin\Omega$, $\tan(i/2)\cos\Omega$, respectively
$\boldsymbol{r}, \dot{\boldsymbol{r}}$	= position and velocity vectors, km and km/s
s_{F_0}, c_{F_0}	= $\sin F_0$, $\cos F_0$, etc.
$\hat{u}$	= unit vector along thrust vector
θ^*	= true anomaly, rad
λ	= mean longitude, $M + \omega + \Omega$, rad
μ	= Earth gravity constant, 398601.3 km^3/s^2

Introduction

THE use of the classical and modern optimization methods in solving low-thrust optimal transfer and rendezvous problems in Earth orbit,[1–4] interplanetary flight,[5–7] and even within the context of the restricted three-body problem[8,9] has been adopted by many contributors with a preference for analytic methods. Scheel and Conway[2] solved the minimum-time problem by using a direct-transcription approach, whereas Kluever and Pierson[3] adopted a combined direct/indirect approach to solve minimum-fuel Earth–moon low-thrust transfer trajectories involving coast arcs and using the restricted circular three-body model. In problems involving two active spacecraft, Coverstone-Carroll and Prussing[4] solved the cooperative power-limited linearized rendezvous problem analytically by further enforcing propellant constraints for a more realistic simulation. The purpose of this paper is not to solve a specific transfer problem or to compare the advantages and disadvantages of selecting the indirect method as opposed to the more commonly used direct method involving collocation and nonsingular programming techniques. Its purpose is rather to provide theoretical insight into the mathematical formulations of optimal transfer and to present the analysis and derivations that lead to the formulation of the optimal transfer problem using a set of fundamental nonsingular orbit elements where the sixth element is an epoch element in the form of the epoch eccentric longitude. The epoch formulations are truly fundamental formulations and form the basis from which the current time formulations are readily derived. The state and adjoint variables, in addition, remain constant in the absence of any perturbation when epoch formulations are used. In an effort to mutually validate the various formulations using various sets of equinoctial elements and in particular the various sets of the adjoint differential equations that are derived with different assumptions, this paper derives the epoch eccentric longitude formulation also showing how it is related mathematically to the corresponding current time eccentric longitude formulation. Simple examples of transfer using continuous constant acceleration are used to compare the time histories of the Lagrange multipliers of both formulations. Furthermore, the constancy of the Hamiltonian in the current time formulation is replaced by the constancy of a certain function that defines the transversality condition for minimum-time problems in this epoch formulation. The Hamiltonian of this epoch formulation is not constant but time varying instead, such that the numerical check, in accepting a converged solution, must rely now on the constancy of the aforementioned transversality function.

A minor drawback of the epoch formulations consist of the appearance of the time, which is the independent variable, explicitly in the state and adjoint equations. This explicit appearance of the time is a nuisance only for applications involving planetary motions where the state vector is integrated over a very long span. Furthermore, the epoch formulations cannot avoid the added computations needed to solve for the eccentric longitude from Kepler's equation through iteration. This avoidance is possible only by considering either the current time eccentric longitude or the current time true longitude as the sixth or fast variable and by expressing the right-hand side of each of the six state equations in terms of F or L, respectively. Numerical comparisons within the context of

Received 30 June 1998; revision received 30 October 1998; accepted for publication 2 November 1998.

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averaging between epoch and current time formulations have not yet been carried out to determine whether some formulations are numerically more robust than others. Additional comparisons are necessary when solving minimum-fuel problems involving coast arcs and/or when considering additional perturbations such as due to J_2 or third-body gravity as well as natural eclipsing during a given transfer. For Earth orbiting applications, nonsingular elements are used[10–14] instead of the classical elements because they do not exhibit any singularities for zero eccentricity and zero inclination.

The full set of governing equations to solve precision integrated optimized transfer problems for the current mean longitude,[12,15] epoch mean longitude,[13] current eccentric longitude,[14] and true longitude[16] formulations have been derived. The derivation of the equations of motion for the epoch eccentric longitude formulation is presented here by direct transformation of the Poisson brackets matrix that corresponds to the epoch mean longitude formulation. As already stated, this particular orbit element is considered a natural element because it is perturbation dependent only and, therefore, stays constant in the absence of any thrust perturbation like the first five elements. Precision integration is used to solve the corresponding two-point boundary-value problem by the iterative shooting method and a quasi-Newton scheme. The mathematical derivations are validated by duplicating two examples of minimum-time transfer generated with other nonsingular formulations.

System and Adjoint Differential Equations for the Epoch Eccentric Longitude Formulation

The most straightforward way of deriving the equations of motion for the set (a, h, k, p, q, F_0) is by mapping the matrix of the Poisson brackets that corresponds to the set $(a, h, k, p, q, \lambda_0)$, which is itself mapped from the matrix of the fundamental classical set $(a, e, i, \Omega, \omega, M_0)$. Once the elements of the Poisson matrix are obtained, the partial derivatives of the elements with respect to the velocity vector are derived, after establishing the partial derivatives of the position vector with respect to the same elements. The former partials provide the required equations of motion after multiplication by the thrust acceleration vector itself. The differential equations of the adjoint variables are derived next from the Hamiltonian after establishing the functional dependence of r and F with respect to the element set at hand, the r and F variables appearing in the right-hand sides of the differential equations for the state variables. Finally, a canonical transformation shows how the various multipliers and the Hamiltonian of the sets (a, h, k, p, q, F) and (a, h, k, p, q, F_0) are related. The plots of these multipliers and Hamiltonians are compared in the next section using a common transfer example, further verifying the corresponding mathematical relationships.

Given initial values of the classical orbit elements at time zero, namely, a_0, e_0, i_0, Ω_0, ω_0, and M_0, the eccentric anomaly E_0 is first obtained from Kepler's equation by iteration:

$$M_0 = E_0 - e_0 \sin E_0 \tag{1}$$

such that the equinoctial elements can now be evaluated. The equations of motion as well as the adjoint system of differential equations using λ as the sixth state variable were shown earlier[12,17,18] with

$$\dot{a} = \left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right) \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{h} = \left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

$$\dot{k} = \left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{p} = \left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

$$\dot{q} = \left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{\lambda} = n + \left(\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

given by Eqs. (1–6) of Ref. 17. The velocity partials or the partials of the equinoctial elements with respect to the velocity vector $\dot{\boldsymbol{r}}$ constitute the rows of the so-called 6×3 M matrix such that they are given by Eqs. (7–12) of Ref. 17 directly in terms of the elements, the components of the velocity vector $\dot{\boldsymbol{r}} = \dot{X}_1 \hat{\boldsymbol{f}} + \dot{Y}_1 \hat{\boldsymbol{g}}$, namely, $\dot{X}_1$ and $\dot{Y}_1$, and the components of the position vector $\boldsymbol{r} = X_1 \hat{\boldsymbol{f}} + Y_1 \hat{\boldsymbol{g}}$, namely, X_1 and Y_1, as well as their partial derivatives with respect to h and k.

The position and velocity vector components are written in terms of the equinoctial elements as in Ref. 17, which also shows the partial derivatives $\partial X_1/\partial h$, $\partial X_1/\partial k$ and $\partial Y_1/\partial h$, $\partial Y_1/\partial k$ as a function of the elements, with $\beta = 1/(1+G)$ and $r = a(1 - k c_F - h s_F)$. These partials are derived by varying F and β with respect to h and k with $\partial\beta/\partial h = h\beta^3/(1-\beta)$, $\partial\beta/\partial k = k\beta^3/(1-\beta)$, and $\partial F/\partial h = -(a/r)c_F$, $\partial F/\partial k = (a/r)s_F$. The expressions for X_1 and Y_1 are obtained by direct transformation of the expressions given in terms of the classical elements $X_1 = r\cos(\theta^* + \omega + \Omega)$, $Y_1 = r\sin(\theta^* + \omega + \Omega)$ and using the identities $r c_{\theta^*} = a(c_E - e)$ and $r s_{\theta^*} = a(1-e^2)^{1/2} s_E$ and the definitions $h = e s_{\omega+\Omega}$, $k = e c_{\omega+\Omega}$, $F = E + \omega + \Omega$, and $e = (h^2 + k^2)^{1/2}$. Letting

$$\beta = \frac{1 - (1 - h^2 - k^2)^{\frac{1}{2}}}{h^2 + k^2} = \left[1 + (1 - h^2 - k^2)^{\frac{1}{2}}\right]^{-1}$$

and observing that

$$\frac{k^2 + h^2(1 - h^2 - k^2)^{\frac{1}{2}}}{h^2 + k^2} = 1 - h^2\beta$$

the expression for X_1 is obtained. Similar manipulations lead to the expression for Y_1. Holding a, h, k, and, therefore, β constant and varying only F as a function of time, the expressions for the velocity components $\dot{X}_1$ and $\dot{Y}_1$ are obtained directly from X_1 and Y_1 after making use of $\dot{\lambda} = n = \dot{F} - k c_F \dot{F} - h s_F \dot{F}$, which yields $\dot{F} = na/r$. The epoch mean longitude λ_0 was used as the sixth state variable[13] instead of λ, and this required replacing the equation for the fast variable λ, namely, $\dot{\lambda} = n + (\partial\lambda/\partial\dot{\boldsymbol{r}})\hat{\boldsymbol{u}} f_t$ by

$$\dot{\lambda}_0 = \left(\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{2}$$

where

$$\begin{aligned}\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}} &= n^{-1}a^{-2}\left[-2X_1 + 3\dot{X}_1 t + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{\boldsymbol{f}} \\ &+ n^{-1}a^{-2}\left[-2Y_1 + 3\dot{Y}_1 t + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{\boldsymbol{g}} \\ &+ n^{-1}a^{-2}G^{-1}(qY_1 - pX_1)\hat{\boldsymbol{w}} = M_{61}^0\hat{\boldsymbol{f}} + M_{62}^0\hat{\boldsymbol{g}} + M_{63}^0\hat{\boldsymbol{w}}\end{aligned} \tag{3}$$

When Eq. (2) is integrated numerically, we must evaluate λ from $\lambda = \lambda_0 + nt$, where n is osculating, and solve for F by iteration from Kepler's equation $\lambda = F - k s_F + h c_F$. This calculation is needed because F appears in the right-hand sides of the dynamic as well as adjoint equations as will be shown. The need for this particular iteration was removed by using the eccentric longitude F as the sixth state variable.[10,14] In this case, Eq. (2) or the $\dot{\lambda}$ differential equation is replaced by

$$\dot{F} = \frac{na}{r} + \left(\frac{\partial F}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{4}$$

where $\partial F/\partial \dot{\boldsymbol{r}}$ is given by

$$\frac{\partial F}{\partial \dot{\boldsymbol{r}}} = \frac{a}{r}\left[\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}} + s_F\frac{\partial k}{\partial \dot{\boldsymbol{r}}} - c_F\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right] = M_{61}^F\hat{\boldsymbol{f}} + M_{62}^F\hat{\boldsymbol{g}} + M_{63}^F\hat{\boldsymbol{w}} \tag{5}$$

or in explicit form

$$\begin{aligned}\frac{\partial F}{\partial \dot{\boldsymbol{r}}} &= \frac{1}{nar}\Bigg[\Bigg\{-2X_1 + G(h\beta - s_F)\frac{\partial X_1}{\partial h} \\ &+ G(k\beta - c_F)\frac{\partial X_1}{\partial k} - \beta G(k s_F - h c_F)\frac{\dot{X}_1}{n}\Bigg\}\hat{\boldsymbol{f}} \\ &+ \Bigg\{-2Y_1 + G(h\beta - s_F)\frac{\partial Y_1}{\partial h} + G(k\beta - c_F)\frac{\partial Y_1}{\partial k} \\ &- \beta G(k s_F - h c_F)\frac{\dot{Y}_1}{n}\Bigg\}\hat{\boldsymbol{g}} + \frac{r}{aG}(qY_1 - pX_1)\hat{\boldsymbol{w}}\Bigg]\end{aligned} \tag{6}$$

The adjoint differential equations were developed for the preceding three formulations,[12–14] where the sixth state variable is chosen, respectively, as the mean longitude, the epoch mean longitude, and the eccentric longitude. All three formulations make use of the same second-order partial derivative of X_1 that appears in the description of the adjoint equations. This particular partial shown in Eq. (A48) of the Appendix of the present paper was written erroneously in Refs. 12, 13, 17, and 18 as

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - h) + \frac{a}{r}s_F c_F\right]$$

which had the unfortunate effect of introducing a small error in some of the examples shown there. In this paper, we consider F_0 as the fast variable, or rather sixth state variable, such that the λ equation as well as Eqs. (2) and (4) are now replaced by

$$\dot{F}_0 = \left(\frac{\partial F_0}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{7}$$

where

$$\begin{aligned}\frac{\partial F_0}{\partial \dot{\boldsymbol{r}}} &= n^{-1}a^{-2}\frac{a_0}{r_0}\left[-2X_1 + 3\dot{X}_1 t + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{\boldsymbol{f}} \\ &+ n^{-1}a^{-2}\frac{a_0}{r_0}\left[-2Y_1 + 3\dot{Y}_1 t + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{\boldsymbol{g}} \\ &+ n^{-1}a^{-2}G^{-1}\frac{a_0}{r_0}(qY_1 - pX_1)\hat{\boldsymbol{w}}\end{aligned} \tag{8}$$

This particular form was derived[11] by also showing how $\partial\lambda_0/\partial\dot{\boldsymbol{r}}$ and $\partial F_0/\partial\dot{\boldsymbol{r}}$ are related. From Kepler's equation evaluated at epoch, we have

$$\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0} \tag{9}$$

$$\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}} = \frac{\partial F_0}{\partial \dot{\boldsymbol{r}}}\left[1 - k_0 c_{F_0} - h_0 s_{F_0}\right] = \frac{r_0}{a_0}\frac{\partial F_0}{\partial \dot{\boldsymbol{r}}} \tag{10}$$

Equation (10) shows that

$$\frac{\partial F_0}{\partial \dot{\boldsymbol{r}}} = \frac{a_0}{r_0}\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}} = \frac{a_0}{r_0}M^0_{61}\hat{\boldsymbol{f}} + \frac{a_0}{r_0}M^0_{62}\hat{\boldsymbol{g}} + \frac{a_0}{r_0}M^0_{63}\hat{\boldsymbol{w}} \tag{11}$$

The Poisson brackets of the fundamental equinoctial elements set $(a, h, k, \lambda_0, p, q)$ are obtained by direct transformation of the brackets of the fundamental classical set $(a, e, i, M_0, \Omega, \omega)$. The transformed matrix is given by

$$[(a_\lambda, a_\mu)] = \frac{1}{na^2}\begin{bmatrix} 0 & 0 & 0 & -2a & 0 & 0 \\ & 0 & -G & \dfrac{hG}{F'} & \dfrac{-kpK}{2G} & \dfrac{-kqK}{2G} \\ & & 0 & \dfrac{kG}{F'} & \dfrac{hpK}{2G} & \dfrac{hqK}{2G} \\ & & & 0 & \dfrac{-pK}{2G} & \dfrac{-qK}{2G} \\ & & & & 0 & \dfrac{-K^2}{4G} \\ -\text{sym} & & & & & 0 \end{bmatrix}$$

with $F' = 1 + G = 1/\beta$. The Poisson brackets of the set (a, h, k, F_0, p, q) are obtained by applying the transformation equation

$$[(p_\alpha, p_\beta)] = \left[\frac{\partial p_\alpha}{\partial a_\lambda}\right][a_\lambda, a_\mu]\left[\frac{\partial p_\beta}{\partial a_\mu}\right]^T$$

where the mapping matrix $[\partial p_\alpha/\partial a_\lambda]$ is given by

$$\left[\frac{\partial p_\alpha}{\partial a_\lambda}\right] = \begin{bmatrix} \dfrac{\partial a}{\partial a} & \dfrac{\partial a}{\partial h} & \dfrac{\partial a}{\partial k} & \dfrac{\partial a}{\partial \lambda_0} & \dfrac{\partial a}{\partial p} & \dfrac{\partial a}{\partial q} \\ \dfrac{\partial h}{\partial a} & \dfrac{\partial h}{\partial h} & \dfrac{\partial h}{\partial k} & \dfrac{\partial h}{\partial \lambda_0} & \dfrac{\partial h}{\partial p} & \dfrac{\partial h}{\partial q} \\ \dfrac{\partial k}{\partial a} & \dfrac{\partial k}{\partial h} & \dfrac{\partial k}{\partial k} & \dfrac{\partial k}{\partial \lambda_0} & \dfrac{\partial k}{\partial p} & \dfrac{\partial k}{\partial q} \\ \dfrac{\partial F_0}{\partial a} & \dfrac{\partial F_0}{\partial h} & \dfrac{\partial F_0}{\partial k} & \dfrac{\partial F_0}{\partial \lambda_0} & \dfrac{\partial F_0}{\partial p} & \dfrac{\partial F_0}{\partial q} \\ \dfrac{\partial p}{\partial a} & \dfrac{\partial p}{\partial h} & \dfrac{\partial p}{\partial k} & \dfrac{\partial p}{\partial \lambda_0} & \dfrac{\partial p}{\partial p} & \dfrac{\partial p}{\partial q} \\ \dfrac{\partial q}{\partial a} & \dfrac{\partial q}{\partial h} & \dfrac{\partial q}{\partial k} & \dfrac{\partial q}{\partial \lambda_0} & \dfrac{\partial q}{\partial p} & \dfrac{\partial q}{\partial q} \end{bmatrix}$$

From Kepler's equation at epoch,

$$\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0}$$

$$\frac{\partial \lambda_0}{\partial \lambda_0} = 1 = \frac{\partial F_0}{\partial \lambda_0}\left(1 - k_0 c_{F_0} - h_0 s_{F_0}\right)$$

such that $\partial F_0/\partial\lambda_0 = a_0/r_0$. Because F_0 is now an orbital element and is, therefore, independent of the elements a, h, k, p, and q, its partials with respect to these five elements are equal to zero, and the preceding matrix is then purely diagonal:

$$\left[\frac{\partial p_\alpha}{\partial a_\lambda}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \dfrac{a_0}{r_0} & 0 & 0 \\ & & & & 1 & 0 \\ & & 0 & & & 1 \end{bmatrix}$$

Because $[\partial p_\beta/\partial a_\mu]^T = [\partial p_\alpha/\partial a_\lambda]^T$, the matrix of the Poisson brackets of the set (a, h, k, F_0, p, q) reduces to

$$[(p_\alpha, p_\beta)] = \frac{1}{na^2}\begin{bmatrix} 0 & 0 & 0 & -2a\dfrac{a_0}{r_0} & 0 & 0 \\ & 0 & -G & Gh\beta\dfrac{a_0}{r_0} & \dfrac{-kpK}{2G} & \dfrac{-kqK}{2G} \\ & & 0 & Gk\beta\dfrac{a_0}{r_0} & \dfrac{hpK}{2G} & \dfrac{hqK}{2G} \\ & & & 0 & \dfrac{-pK}{2G}\dfrac{a_0}{r_0} & \dfrac{-qK}{2G}\dfrac{a_0}{r_0} \\ & & & & 0 & \dfrac{-K^2}{4G} \\ -\text{sym} & & & & & 0 \end{bmatrix}$$

The partials of the elements of the set (a, h, k, F_0, p, q) with respect to the velocity vector $\dot{\boldsymbol{r}}$ can now be written as

$$\frac{\partial a}{\partial \dot{\boldsymbol{r}}} = -(a, F_0)\frac{\partial \boldsymbol{r}}{\partial F_0}$$

$$\frac{\partial h}{\partial \dot{\boldsymbol{r}}} = -(h, k)\frac{\partial \boldsymbol{r}}{\partial k} - (h, F_0)\frac{\partial \boldsymbol{r}}{\partial F_0} - (h, p)\frac{\partial \boldsymbol{r}}{\partial p} - (h, q)\frac{\partial \boldsymbol{r}}{\partial q}$$

$$\frac{\partial k}{\partial \dot{\boldsymbol{r}}} = -(k, h)\frac{\partial \boldsymbol{r}}{\partial h} - (k, F_0)\frac{\partial \boldsymbol{r}}{\partial F_0} - (k, p)\frac{\partial \boldsymbol{r}}{\partial p} - (k, q)\frac{\partial \boldsymbol{r}}{\partial q}$$

$$\frac{\partial F_0}{\partial \dot{\boldsymbol{r}}} = -(F_0, a)\frac{\partial \boldsymbol{r}}{\partial a} - (F_0, h)\frac{\partial \boldsymbol{r}}{\partial h} - (F_0, k)\frac{\partial \boldsymbol{r}}{\partial k} - (F_0, p)\frac{\partial \boldsymbol{r}}{\partial p} - (F_0, q)\frac{\partial \boldsymbol{r}}{\partial q}$$

$$\frac{\partial p}{\partial \dot{\boldsymbol{r}}} = -(p, h)\frac{\partial \boldsymbol{r}}{\partial h} - (p, k)\frac{\partial \boldsymbol{r}}{\partial k} - (p, F_0)\frac{\partial \boldsymbol{r}}{\partial F_0} - (p, q)\frac{\partial \boldsymbol{r}}{\partial q}$$

$$\frac{\partial q}{\partial \dot{\boldsymbol{r}}} = -(q, h)\frac{\partial \boldsymbol{r}}{\partial h} - (q, k)\frac{\partial \boldsymbol{r}}{\partial k} - (q, F_0)\frac{\partial \boldsymbol{r}}{\partial F_0} - (q, p)\frac{\partial \boldsymbol{r}}{\partial p}$$

The partials of $\mathbf{r}$ with respect to the elements are obtained by allowing for the variation of F with respect to the same elements, with $\partial F/\partial a = -\frac{3}{2}(n/r)t$, $\partial F/\partial h = -(a/r)c_F$, and $\partial F/\partial k = (a/r)s_F$. From $\lambda = F - ks_F + hc_F$, partial $\partial\lambda/\partial F = r/a$ such that from $\partial\lambda/\partial a = (\partial\lambda/\partial F)(\partial F/\partial a) = (\partial n/\partial a)t = -\frac{3}{2}(n/a)t$, it follows that $\partial\lambda/\partial a = -\frac{3}{2}(n/r)t$. In a similar way, from $\partial\lambda/\partial h = 0 = \partial F/\partial h - kc_F\partial F/\partial h + c_F - hs_F\partial F/\partial h$, it follows that $\partial F/\partial h = -(a/r)c_F$, and in an equivalent way, from $\partial\lambda/\partial k = 0$, it follows that $\partial F/\partial k = (a/r)s_F$. Finally, from

$$\frac{\partial \mathbf{r}}{\partial \lambda} = \left(\frac{\partial \mathbf{r}}{\partial F}\right)\left(\frac{\partial F}{\partial \lambda}\right) = \left(\frac{\partial X_1}{\partial F}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial F}\hat{\mathbf{g}}\right)\frac{\partial F}{\partial \lambda}$$

and because from

$$\frac{\partial \lambda}{\partial \lambda} = 1 = \frac{\partial F}{\partial \lambda}(1 - kc_F - hs_F) = \frac{\partial F}{\partial \lambda}\frac{r}{a}$$

we have $\partial F/\partial\lambda = a/r$, the partial $\partial\mathbf{r}/\partial\lambda$ reduces to

$$\frac{\partial \mathbf{r}}{\partial \lambda} = \frac{\dot{X}_1}{n}\hat{\mathbf{f}} + \frac{\dot{Y}_1}{n}\hat{\mathbf{g}} = \frac{\dot{\mathbf{r}}}{n}$$

This expression is now used in

$$\frac{\partial \mathbf{r}}{\partial F_0} = \frac{\partial \mathbf{r}}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0} = \frac{\partial \mathbf{r}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0}\frac{\partial \mathbf{r}}{\partial F_0} = \frac{\partial \mathbf{r}}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0} = \frac{\partial \mathbf{r}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0}$$

because $\partial\lambda/\partial\lambda_0 = 1$ and from $\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0}$, the partial $\partial\lambda_0/\partial F_0 = r_0/a_0$. Therefore,

$$\frac{\partial \mathbf{r}}{\partial a} = \frac{\partial X_1}{\partial a}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial a}\hat{\mathbf{g}} = \left(\frac{X_1}{a} - \frac{3}{2}\frac{t}{a}\dot{X}_1\right)\hat{\mathbf{f}} + \left(\frac{Y_1}{a} - \frac{3}{2}\frac{t}{a}\dot{Y}_1\right)\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial h} = \frac{\partial X_1}{\partial h}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial h}\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial k} = \frac{\partial X_1}{\partial k}\hat{\mathbf{f}} + \frac{\partial Y_1}{\partial k}\hat{\mathbf{g}}$$

$$\frac{\partial \mathbf{r}}{\partial F_0} = \frac{r_0}{a_0}\left(\frac{\dot{X}_1}{n}\hat{\mathbf{f}} + \frac{\dot{Y}_1}{n}\hat{\mathbf{g}}\right)$$

where the partials $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$ are given in Ref. 17, as stated earlier. The transformation from the inertial $(\hat{x}, \hat{y}, \hat{z})$ frame to the equinoctial $(\hat{\mathbf{f}}, \hat{\mathbf{g}}, \hat{\mathbf{w}})$ frame is given by

$$\begin{bmatrix}\hat{f}\\ \hat{g}\\ \hat{w}\end{bmatrix} = \begin{bmatrix} c_\Omega^2 + c_i s_\Omega^2 & s_\Omega c_\Omega - s_\Omega c_\Omega c_i & -s_\Omega s_i \\ s_\Omega c_\Omega - s_\Omega c_\Omega c_i & s_\Omega^2 + c_i c_\Omega^2 & s_i c_\Omega \\ s_\Omega s_i & -c_\Omega s_i & c_i \end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$

$$= \frac{1}{K}\begin{bmatrix} 1 - p^2 + q^2 & 2pq & -2p \\ 2pq & 1 + p^2 - q^2 & 2q \\ 2p & -2q & 1 - p^2 - q^2 \end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$

The partials of the unit vectors $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ with respect to p and q are readily obtained to yield

$$\frac{\partial \mathbf{r}}{\partial p} = X_1\frac{\partial \hat{\mathbf{f}}}{\partial p} + Y_1\frac{\partial \hat{\mathbf{g}}}{\partial p} = \frac{2}{K}[q(Y_1\hat{\mathbf{f}} - X_1\hat{\mathbf{g}}) - X_1\hat{\mathbf{w}}]$$

$$\frac{\partial \mathbf{r}}{\partial q} = X_1\frac{\partial \hat{\mathbf{f}}}{\partial q} + Y_1\frac{\partial \hat{\mathbf{g}}}{\partial q} = \frac{2}{K}[p(X_1\hat{\mathbf{g}} - Y_1\hat{\mathbf{f}}) + Y_1\hat{\mathbf{w}}]$$

The partials of the elements a, h, k, p, q, and F_0 with respect to the velocity vector take the form given in Eqs. (7–11) of Ref. 17 and Eq. (8) here. Because we already have shown an expression[13] for $\partial\lambda_0/\partial\dot{\mathbf{r}}$, we shall keep $\partial F_0/\partial\dot{\mathbf{r}}$, as given in Eq. (11) here, to make use of most of the partial derivatives of $\partial\lambda_0/\partial\dot{\mathbf{r}}$ with respect to the elements developed in Ref. 13. This will save us some additional coding effort. Now once E_0 has been computed from Eq. (1), we get $F_0 = E_0 + \omega_0 + \Omega_0$, $h_0 = e_0\sin(\omega_0 + \Omega_0)$, and $k_0 = e_0\cos(\omega_0 + \Omega_0)$. The values of h_0, k_0 and, therefore, $e_0 = (h_0^2 + k_0^2)^{1/2}$ remain constant. However, F_0 is being integrated and, therefore, E_0 will vary according to $E_0 = F_0 - \tan^{-1}(h_0/k_0)$. Now λ_0 is evaluated from $\lambda_0 = E_0 - e_0\sin E_0 + \tan^{-1}(h_0/k_0)$ because, in the definition of $\lambda = M + \omega + \Omega$, the equation $M = E - e\sin E$ is used as well as the inverse transformation relations for ω and Ω, namely, $\omega = \tan^{-1}(h/k) - \tan^{-1}(p/q)$ and $\Omega = \tan^{-1}(p/q)$. The current mean longitude is now evaluated from $\lambda = \lambda_0 + nt$, where n is also osculating, and F is solved for through iteration from Kepler's equation $\lambda = F - ks_F + hc_F$. Although a_0 remains constant, $r_0 = a_0(1 - k_0 c_{F_0} - h_0 s_{F_0})$ is a function of F_0 such that the factor a_0/r_0 in Eq. (11) is a function of only one element, namely, F_0. The following partial is then needed to evaluate $\dot{\lambda}_{F_0}$, the differential equation for the adjoint to F_0:

$$\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}\right) = \left(\frac{a_0}{r_0}\right)^2\left(h_0 c_{F_0} - k_0 s_{F_0}\right) \tag{12}$$

The Hamiltonian for the F_0 formulation is given by

$$\begin{aligned} H = \{&\lambda_a(M_{11}u_f + M_{12}u_g + M_{13}u_w) + \lambda_h(M_{21}u_f \\ &+ M_{22}u_g + M_{23}u_w) + \lambda_k(M_{31}u_f + M_{32}u_g + M_{33}u_w) \\ &+ \lambda_p(M_{41}u_f + M_{42}u_g + M_{43}u_w) + \lambda_q(M_{51}u_f + M_{52}u_g \\ &+ M_{53}u_w) + \lambda_{F_0}[(a_0/r_0)M_{61}^0 u_f + (a_0/r_0)M_{62}^0 u_g \\ &+ (a_0/r_0)M_{63}^0 u_w]\} f_t \end{aligned} \tag{13}$$

or, in compact form, by

$$H = \boldsymbol{\lambda}_z^T\dot{\mathbf{z}} = \boldsymbol{\lambda}_z^T M^{F_0}(\mathbf{z}, F) f_t\hat{\mathbf{u}} \tag{14}$$

where $\mathbf{z} = (a\ h\ k\ p\ q\ F_0)^T$ represents the state vector and where the 6×3 matrix M^{F_0} is now given by

$$M^{F_0} = \begin{bmatrix} \left(\frac{\partial a}{\partial \dot{\mathbf{r}}}\right)^T \\ \left(\frac{\partial h}{\partial \dot{\mathbf{r}}}\right)^T \\ \left(\frac{\partial k}{\partial \dot{\mathbf{r}}}\right)^T \\ \left(\frac{\partial p}{\partial \dot{\mathbf{r}}}\right)^T \\ \left(\frac{\partial q}{\partial \dot{\mathbf{r}}}\right)^T \\ \left(\frac{\partial F_0}{\partial \dot{\mathbf{r}}}\right)^T \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \\ M_{41} & M_{42} & M_{43} \\ M_{51} & M_{52} & M_{53} \\ \frac{a_0}{r_0}M_{61}^0 & \frac{a_0}{r_0}M_{62}^0 & \frac{a_0}{r_0}M_{63}^0 \end{bmatrix} \tag{15}$$

The Euler–Lagrange or adjoint differential equations are written as

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial \mathbf{z}} = -\boldsymbol{\lambda}_z^T\frac{\partial M^{F_0}}{\partial \mathbf{z}} f_t\hat{\mathbf{u}} \tag{16}$$

Considering continuous constant acceleration, the thrust direction $\hat{\mathbf{u}} = (u_f, u_g, u_w)$ is optimized by selecting $\hat{\mathbf{u}}$ parallel to the vector

$\dot{\lambda}_z^T M^{F_0}(z, F)$. The adjoint equations are given in expanded form by

$$\dot{\lambda}_a = -\begin{pmatrix}\lambda_a & \lambda_h & \lambda_k & \lambda_p & \lambda_q & \lambda_{F_0}\end{pmatrix}\frac{\partial M^{F_0}}{\partial a} f_t \hat{u} = -\frac{\partial H}{\partial a}$$

$$= \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial a}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial a}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial a}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial a}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial a}u_i\right) - \lambda_{F_0}\left(\frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial a}u_f + \frac{a_0}{r_0}\frac{\partial M_{62}^0}{\partial a}u_g + \frac{a_0}{r_0}\frac{\partial M_{63}^0}{\partial a}u_w\right)\right] f_t \tag{17}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial h}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial h}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial h}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial h}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial h}u_i\right) - \lambda_{F_0}\left(\frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial h}u_f + \frac{a_0}{r_0}\frac{\partial M_{62}^0}{\partial h}u_g + \frac{a_0}{r_0}\frac{\partial M_{63}^0}{\partial h}u_w\right)\right] f_t \tag{18}$$

$$\dot{\lambda}_k = -\frac{\partial H}{\partial k} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial k}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial k}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial k}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial k}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial k}u_i\right) - \lambda_{F_0}\left(\frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial k}u_f + \frac{a_0}{r_0}\frac{\partial M_{62}^0}{\partial k}u_g + \frac{a_0}{r_0}\frac{\partial M_{63}^0}{\partial k}u_w\right)\right] f_t \tag{19}$$

$$\dot{\lambda}_p = -\frac{\partial H}{\partial p} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial p}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial p}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial p}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial p}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial p}u_i\right) - \lambda_{F_0}\left(\frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial p}u_f + \frac{a_0}{r_0}\frac{\partial M_{62}^0}{\partial p}u_g + \frac{a_0}{r_0}\frac{\partial M_{63}^0}{\partial p}u_w\right)\right] f_t \tag{20}$$

$$\dot{\lambda}_q = -\frac{\partial H}{\partial q} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial q}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial q}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial q}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial q}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial q}u_i\right) - \lambda_{F_0}\left(\frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial q}u_f + \frac{a_0}{r_0}\frac{\partial M_{62}^0}{\partial q}u_g + \frac{a_0}{r_0}\frac{\partial M_{63}^0}{\partial q}u_w\right)\right] f_t \tag{21}$$

$$\dot{\lambda}_{F_0} = -\frac{\partial H}{\partial F_0} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial F_0}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial F_0}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial F_0}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial F_0}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial F_0}u_i\right) - \lambda_{F_0}\left(\frac{\partial\left((a_0/r_0)M_{61}^0\right)}{\partial F_0}u_f + \frac{\partial\left((a_0/r_0)M_{62}^0\right)}{\partial F_0}u_g + \frac{\partial\left((a_0/r_0)M_{63}^0\right)}{\partial F_0}u_w\right)\right] f_t \tag{22}$$

The last three partials in Eq. (22) can be written as

$$\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}M_{61}^0\right) = \frac{a_0}{r_0}\frac{\partial M_{61}^0}{\partial F_0} + M_{61}^0\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}\right) = \frac{a_0}{r_0}\frac{r_0}{a_0}\frac{\partial M_{61}^0}{\partial \lambda_0} + M_{61}^0\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}\right) = \frac{\partial M_{61}^0}{\partial \lambda_0} + \left(\frac{a_0}{r_0}\right)^2\left(h_0 c_{F_0} - k_0 s_{F_0}\right)M_{61}^0 \tag{23}$$

$$\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}M_{62}^0\right) = \frac{\partial M_{62}^0}{\partial \lambda_0} + \left(\frac{a_0}{r_0}\right)^2\left(h_0 c_{F_0} - k_0 s_{F_0}\right)M_{62}^0 \tag{24}$$

$$\frac{\partial}{\partial F_0}\left(\frac{a_0}{r_0}M_{63}^0\right) = \frac{\partial M_{63}^0}{\partial \lambda_0} + \left(\frac{a_0}{r_0}\right)^2\left(h_0 c_{F_0} - k_0 s_{F_0}\right)M_{63}^0 \tag{25}$$

where use is made of Eq. (12) and of the following identities:

$$\frac{\partial M_{ij}}{\partial F_0} = \frac{\partial M_{ij}}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0} = \frac{r_0}{a_0}\frac{\partial M_{ij}}{\partial \lambda_0} \qquad i = 1, 6; \qquad j = 1, 3 \tag{26}$$

because, from $\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0}$, we have $\partial\lambda_0/\partial F_0 = 1 - k_0 c_{F_0} - h_0 s_{F_0} = r_0/a_0$. It is also true that $\partial F_0/\partial\lambda_0 = a_0/r_0$. Now the following partial derivatives must be used when the derivations of the $\partial M^{F_0}/\partial z$ partial derivatives shown in the Appendix are carried out, namely,

$$\frac{\partial r}{\partial a} = \frac{r}{a} - \frac{3}{2}\frac{nat}{r}(k s_F - h c_F) \tag{27}$$

$$\frac{\partial r}{\partial h} = \frac{a^2}{r}(h - s_F) \tag{28}$$

$$\frac{\partial r}{\partial k} = \frac{a^2}{r}(k - c_F) \tag{29}$$

$$\frac{\partial r}{\partial F} = a(k s_F - h c_F) \tag{30}$$

$$\frac{\partial F}{\partial a} = -\frac{3}{2}\frac{nt}{r} \tag{31}$$

$$\frac{\partial F}{\partial h} = -\frac{a}{r}c_F \tag{32}$$

$$\frac{\partial F}{\partial k} = \frac{a}{r}s_F \tag{33}$$

$$\frac{\partial F}{\partial F_0} = \frac{r_0}{a_0}\frac{a}{r} = \frac{r_0}{a_0}\frac{\partial F}{\partial \lambda_0} \tag{34}$$

These expressions can be shown to be true from $r = a(1 - k c_F - h s_F)$ by first writing the differential

$$\mathrm{d}r = (1 - k c_F - h s_F)\,\mathrm{d}a - a c_F\,\mathrm{d}k - a s_F\,\mathrm{d}h + a(k s_F - h c_F)\,\mathrm{d}F \tag{35}$$

From $\lambda = \lambda_0 + nt = F - k s_F + h c_F$, it follows that

$$\mathrm{d}\lambda_0 + n\,\mathrm{d}t + t\,\mathrm{d}n = (1 - k c_F - h s_F)\,\mathrm{d}F - s_F\,\mathrm{d}k + c_F\,\mathrm{d}h$$

and because $\mathrm{d}n = -\frac{3}{2}n/a\,\mathrm{d}a$,

$$\mathrm{d}F = [\mathrm{d}\lambda_0 + n\,\mathrm{d}t - \tfrac{3}{2}(nt/a)\,\mathrm{d}a + s_F\,\mathrm{d}k - c_F\,\mathrm{d}h](a/r) \tag{36}$$

However, from Kepler's equation at epoch with h_0 and k_0 fixed, $\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0}$,

$$\mathrm{d}\lambda_0 = \left(1 - k_0 c_{F_0} - h_0 s_{F_0}\right)\mathrm{d}F_0 = (r_0/a_0)\,\mathrm{d}F_0 \tag{37}$$

which, when replaced in Eq. (36), yields

$$dF = [(r_0/a_0)\,dF_0 + n\,dt - \tfrac{3}{2}(nt/a)\,da + s_F\,dk - c_F\,dh](a/r) \tag{38}$$

The partials in Eqs. (31–34) are now readily obtained from Eq. (38). If we now use dF of Eq. (38) in Eq. (35) for dr, the partials in Eqs. (27–30) also are obtained. The last partial in Eq. (34) also can be obtained from

$$\lambda = \lambda_0 + nt = F - ks_F + hc_F = F_0 - k_0 s_{F_0} + h_0 c_{F_0} + nt$$

such that

$$\frac{\partial F}{\partial F_0} - kc_F\frac{\partial F}{\partial F_0} - hs_F\frac{\partial F}{\partial F_0} = \frac{\partial F_0}{\partial F_0} - k_0 c_{F_0} - h_0 s_{F_0}$$

or

$$(1 - kc_F - hs_F)\frac{\partial F}{\partial F_0} = 1 - k_0 c_{F_0} - h_0 s_{F_0} = \frac{r_0}{a_0}$$

and finally, $\partial F/\partial F_0 = (r_0/a_0)(a/r)$. If we now use the canonicity condition for the transformation between $(a\ h\ k\ p\ q\ F)$ and $(a\ h\ k\ p\ q\ F_0)$, we then can show how the various multipliers used in the F formulation[14] relate to the multipliers used in the F_0 formulation. Using F and F_0 superscripts, the canonicity condition is given by

$$\begin{aligned}\lambda_a^F\,da + \lambda_h^F\,dh + \lambda_k^F\,dk + \lambda_p^F\,dp + \lambda_q^F\,dq + \lambda_F\,dF - H^F\,dt \\ = \lambda_a^{F_0}\,da + \lambda_h^{F_0}\,dh + \lambda_k^{F_0}\,dk + \lambda_p^{F_0}\,dp + \lambda_q^{F_0}\,dq \\ + \lambda_{F_0}\,dF_0 - H^{F_0}\,dt\end{aligned} \tag{39}$$

Replacing dF from Eq. (38) in the preceding condition and identifying identical terms yields

$$\lambda_a^{F_0} = \lambda_a^F - \tfrac{3}{2}(nt/a)(a/r)\lambda_F \tag{40}$$

$$\lambda_h^{F_0} = \lambda_h^F - (a/r)c_F\lambda_F \tag{41}$$

$$\lambda_k^{F_0} = \lambda_k^F + (a/r)s_F\lambda_F \tag{42}$$

$$\lambda_p^{F_0} = \lambda_p^F \tag{43}$$

$$\lambda_q^{F_0} = \lambda_q^F \tag{44}$$

$$\lambda_{F_0} = (a/r)(r_0/a_0)\lambda_F \tag{45}$$

$$H^{F_0} = H^F - (na/r)\lambda_F \tag{46}$$

Transversality Condition for Minimum-Time Rendezvous and Examples of Free-Free Minimum-Time Transfer

This section shows how the transversality condition at the final unknown time for minimum-time problems can effectively replace the Hamiltonian constancy check corresponding to the current longitude formulations in accepting a converged transfer trajectory as being truly optimal. The function that defines the transversality condition is, thus, shown to remain constant in this new formulation even though the Hamiltonian is no longer constant. The optimality check is, thus, still possible by verifying the constancy of this function instead.

As in Refs. 13 and 14, rendezvous time is minimized by maximizing

$$J = -\int_{t_0}^{t} dt = -(t_f - t_0) \tag{47}$$

The initial values of the six Lagrange multipliers $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$, and $(\lambda_{F_0})_0$, as well as the transfer time t_f, are guessed; starting from initial state variables a_0, h_0, k_0, p_0, q_0, and $(F_0)_0$, the state and adjoint equations are integrated forward until $t = t_f$, using the optimal control

$$u = \frac{\left(\lambda_z^T M^{F_0}\right)^T}{\left|\lambda_z^T M^{F_0}\right|} \tag{48}$$

A quasi-Newton scheme is used to iterate on the initial values of the multipliers, as well as the transfer time, such that the desired terminal state given by a_f, h_f, k_f, p_f, q_f, and F_f is reached and the transversality condition at the unknown final time is satisfied:

$$\left(\frac{\partial\Phi}{\partial t} + \lambda_z^T\dot{z}\right)_{t_f} = 1 \tag{49}$$

with

$$\Phi = \nu\psi = \nu\left[(F_0)_f - k_0 s_{(F_0)_f} + h_0 c_{(F_0)_f} + n_f t_f - \lambda_f\right]$$

because we want to match $\lambda_f = (\lambda_0)_t + n_f t_f$, which is a function of the state variables a and F_0. The constant Lagrange multiplier ν is obtained from the additional condition $(\lambda_{F_0})_f = (\nu(\partial\psi/\partial F_0))_{t_f}$, which yields $(\lambda_{F_0})_f = \nu(1 - k_0 c_{(F_0)_f} - h_0 s_{(F_0)_f}) = \nu(r_0)_f/a_0$, such that

$$\nu = \frac{a_0}{(r_0)_f}(\lambda_{F_0})_f \tag{50}$$

Because $H = \lambda_z^T\dot{z}$, the transversality condition in Eq. (49) yields $\nu(\partial\psi/\partial t) + H_f = 1$, or

$$T_r = \frac{a_0 n_f}{(r_0)_f}(\lambda_{F_0})_f + H_f = 1 \tag{51}$$

From Eqs. (45) and (46), the preceding condition yields $(H^F)_f = 1$, the transversality condition of the F formulation. We can now duplicate the minimum-time transfer example of Refs. 15 and 16 by generating an open-loop trajectory from the given initial orbit shown in Table 1. The initial values of the Lagrange multipliers are identical to the ones corresponding to the eccentric longitude formulation, as can be seen in Eqs. (40–45), because $(\lambda_F)_0 = 0$ for an optimized initial location. This equivalence holds true also between the formulations using the eccentric and mean longitudes,[14] respectively, such that the iterated values of the multipliers shown in Refs. 15 and 16 and in Table 1 can be used to start the numerical integration. The constant acceleration $f_t = 9.8 \times 10^{-5}$ km/s^2 is applied from $t_0 = 0$ to $t_f = 58089.9005$ s to yield the final conditions shown in Table 1, with $H_f = 1.003704684$, closely matching the desired target.

The initial values of the multipliers can be scaled to provide $H_f = 1$ exactly, yielding $(\lambda_a)_0 = 4.657973438$ s/km, $(\lambda_h)_0 = 5.393432977 \times 10^2$ s, $(\lambda_k)_0 = -9.168734810 \times 10^3$ s, $(\lambda_p)_0 = 1.771449217 \times 10^1$ s, $(\lambda_q)_0 = -2.250119870 \times 10^4$ s, and $(\lambda_{F_0})_0 = 0$ s/rad with $H_f = 0.999999998$. It can also be observed that $(a_0/r_0)n(\lambda_{F_0}) + H$ remains constant throughout the transfer, thereby providing a numerical check for accepting a converged trajectory as a truly optimal solution because now H is not constant during the integration. Figure 1 shows the evolution of F and F_0 in time whereas the Hamiltonians H^F of the F formulation

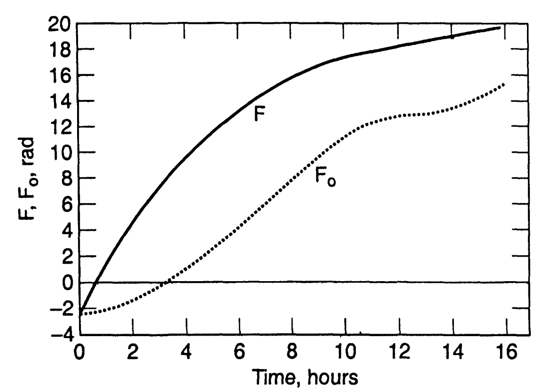

Fig. 1 Eccentric longitude and epoch eccentric longitude time histories during optimal transfer.

Table 1 Transfer parameters

Orbit	Initial	Target	Achieved	Solution	Initial values
a, km	7,000	42,000	42,000.003	$(\lambda_a)_0$, s/km	4.675229762
e	0	10^{-3}	9.986×10^{-4}	$(\lambda_h)_0$, s	541.3413947
i, deg	28.5	1	0.999809	$(\lambda_k)_0$, s	−9,202.702084
Ω, deg	0	0	1.155×10^{-4}	$(\lambda_p)_0$, s	17.78011878
ω, deg	0	0	1.776×10^{-2}	$(\lambda_q)_0$, s	−22,584.55855
M_0, M, deg	$M_0 = -130.3331648$ (optimized)	Free	$M = 46.147008$	$(\lambda_{F_0})_0$, s/rad	0

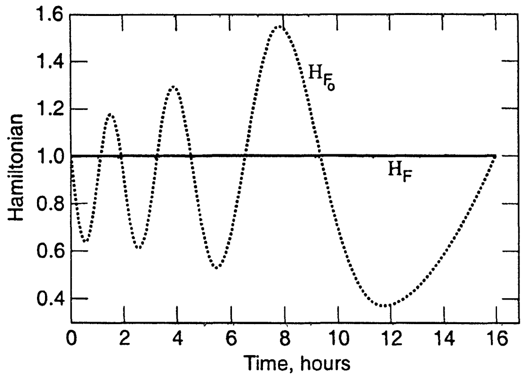

Fig. 2 Evolution of Hamiltonians H^F and H^{F_0} during optimal transfer.

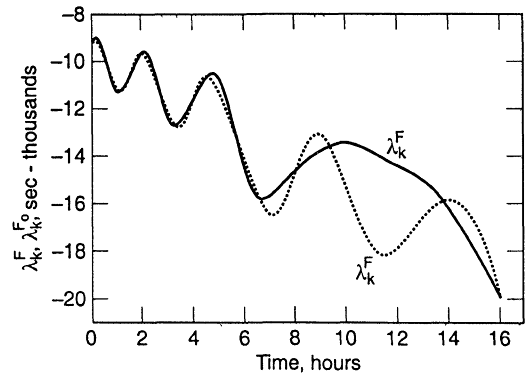

Fig. 5 Evolution of λ_k^F and $\lambda_k^{F_0}$ multipliers during optimal transfer.

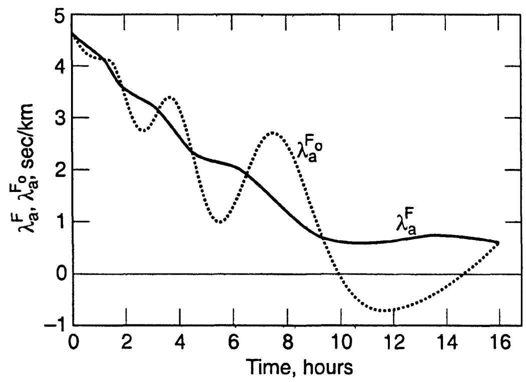

Fig. 3 Evolution of λ_a^F and $\lambda_a^{F_0}$ multipliers during optimal transfer.

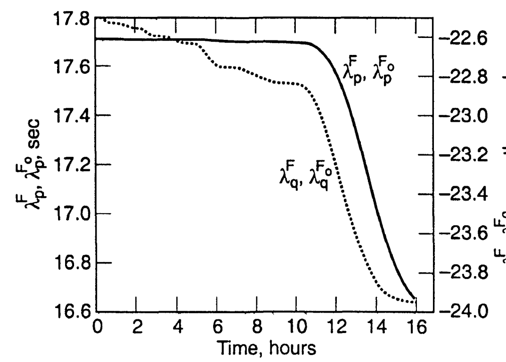

Fig. 6 Evolution of λ_p^F, $\lambda_p^{F_0}$ and λ_q^F, $\lambda_q^{F_0}$ multipliers during optimal transfer.

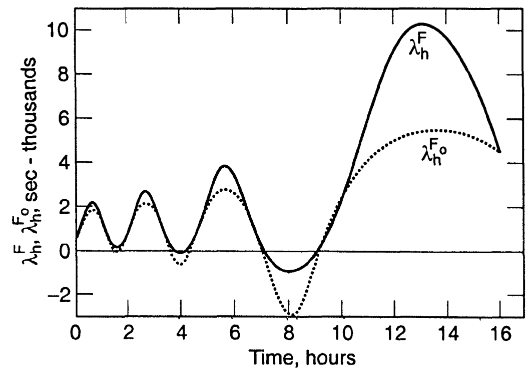

Fig. 4 Evolution of λ_h^F and $\lambda_h^{F_0}$ multipliers during optimal transfer.

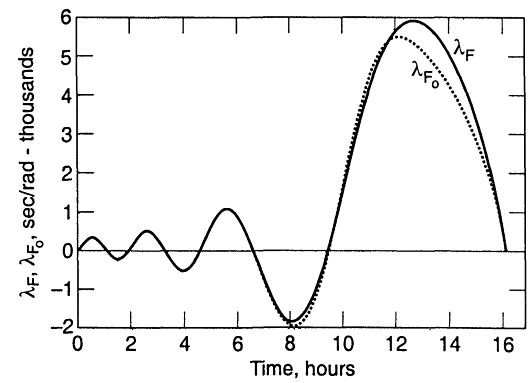

Fig. 7 Evolution of λ_F and λ_{F_0} multipliers during optimal transfer.

and H^{F_0} of the present F_0 formulation are shown in Fig. 2, with $H^F = 1$ throughout and H^{F_0} varying and reaching the value of 1 at the end time. The multipliers' histories are shown in Figs. 3–7 for both the F and F_0 formulations, starting from the same initial values and also ending with equal values in compliance with Eqs. (40–45) due to $(\lambda_F)_f = (\lambda_{F_0})_f = 0$ as verified in Fig. 7. The open-loop runs yield $(\lambda_F)_f = 8.478 \times 10^{-3}$ s/rad and $(\lambda_{F_0})_f = 1.631 \times 10^{-5}$ s/rad, both close to the theoretical value of 0. Furthermore, $H^F = 1.000000015$ throughout the transfer, and the quantity $(a_0/r_0)n\lambda_{F_0} + H = 1.00000000$ remains constant with the relative and absolute integration error controls set to 10^{-9}. The correspondence between the F_0 and the epoch mean longitude formulation λ_0 of Ref. 13 can be shown to be $\lambda_a^{F_0} = \lambda_a^{\lambda_0}$, $\lambda_h^{F_0} = \lambda_h^{\lambda_0}$, $\lambda_k^{F_0} = \lambda_k^{\lambda_0}$, $\lambda_p^{F_0} = \lambda_p^{\lambda_0}$, $\lambda_q^{F_0} = \lambda_q^{\lambda_0}$, $\lambda_{F_0}^{F_0} = (r_0/a_0)\lambda_{\lambda_0}^{\lambda_0}$, and $H^{F_0} = H^{\lambda_0}$. For our initial circular orbit example, $h_0 = k_0 = 0$ such that r_0/a_0 remains constant and equal to 1 and, therefore, $\lambda_{F_0}^{F_0} = \lambda_{\lambda_0}^{\lambda_0}$ throughout the transfer. An open-loop trajectory is generated using the same initial values for the scaled multipliers as the earlier ones and starting from the same optimized initial location with $M_0 = -130.3331648$ deg using the software

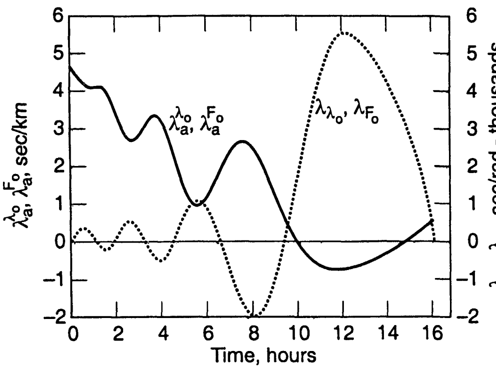

Fig. 8 Evolution of λ_{F_0}, λ_{λ_0}, $\lambda_a^{F_0}$, and $\lambda_a^{\lambda_0}$ multipliers during optimal transfer.

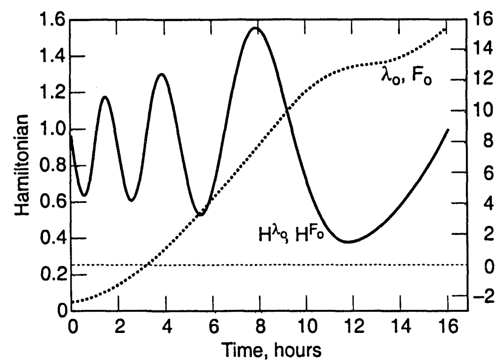

Fig. 11 Evolution of λ_0, F_0, and Hamiltonians H^{λ_0} and H^{F_0} during optimal transfer.

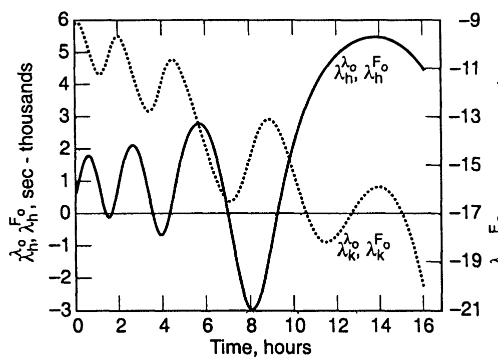

Fig. 9 Evolution of $\lambda_h^{F_0}$, $\lambda_h^{\lambda_0}$, $\lambda_k^{F_0}$, and $\lambda_k^{\lambda_0}$ multipliers during optimal transfer.

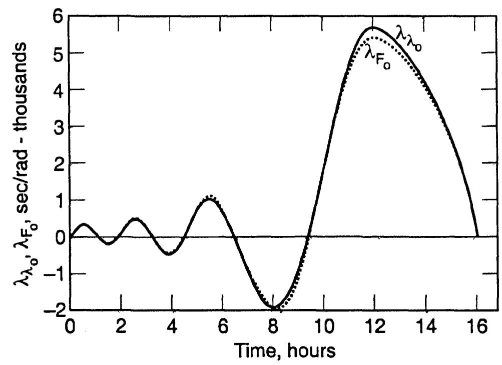

Fig. 12 Evolution of λ_{F_0} and λ_{λ_0} multipliers for transfer with initial eccentricity $e_0 = 0.05$.

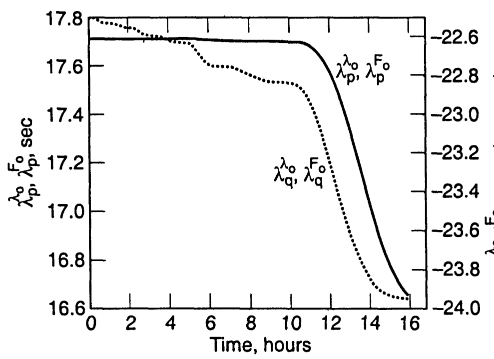

Fig. 10 Evolution of $\lambda_p^{F_0}$, $\lambda_p^{\lambda_0}$, $\lambda_q^{F_0}$, and $\lambda_q^{\lambda_0}$ multipliers during optimal transfer.

of Ref. 13. The evolutions of the various multipliers are shown in Figs. 8–11, with a perfect match between these λ_0 and F_0 epoch formulations. As shown in Fig. 11, both Hamiltonians are varying identically, and both λ_0 and F_0 state variables exhibit the same variations. This can also be seen from Kepler's equation at epoch $\lambda_0 = F_0 - k_0 s_{F_0} + h_0 c_{F_0}$ with $h_0 = k_0 = 0$. It can be seen that the differential equations for the multipliers, Eqs. (17–22), become identical to the corresponding equations of the λ_0 formulation[13] as soon as $e_0 = 0$. Then $r_0/a_0 = 1$, and with $h_0 = k_0 = 0$, Eqs. (23–25) yield $\partial/\partial F_0((a_0/r_0)M_{6i}^0) = \partial M_{6i}^0/\partial\lambda_0$, such that using the expression $\lambda_{F_0}^{F_0} = (r_0/a_0)\lambda_{\lambda_0}^{\lambda_0}$ relating $\lambda_{F_0}^{F_0}$ and $\lambda_{\lambda_0}^{\lambda_0}$, all six pairs of the adjoint differential equations of the F_0 and λ_0 formulations become identical. For this reason, and to further validate the mathematical derivations, we now solve the same minimum-time transfer example except that the initial value of the eccentricity is nonzero, namely, $e_0 = 0.05$, such that $h_0 = 0$ and $k_0 = 0.05$, and use the software of Ref. 12 to determine the initial values of the multipliers, the optimized departure mean anomaly, and the transfer time, as $(\lambda_a)_0 = 4.719028797$ s/km, $(\lambda_h)_0 = 5.490984314 \times 10^2$ s, $(\lambda_k)_0 = -9.651612930 \times 10^3$ s, $(\lambda_p)_0 = -9.379552824 \times 10^1$ s, $(\lambda_q)_0 = -2.270946531 \times 10^4$ s, $M_0 = -130.5130449$ deg, and $t_f = 58560.2470$ s. The two-point boundary-value problem is solved by enforcing the transversality condition $H_f = 1$, which also stays constant in this λ formulation[12] to within 9 decimal places during the integration, indicating thereby a converged trajectory. This solution is run open loop using both the present F_0 formulation and the epoch mean longitude λ_0 formulation[13] to verify the satisfaction of the target conditions, which are closely matched with $a_f = 42{,}000.012$ km, $e_f = 1.00035 \times 10^{-3}$, $i_f = 1.000015$ deg, $\Omega_f = 359.999777$ deg, $\omega_f = 2.3283 \times 10^{-2}$ deg, $M_f = 46.091379$ deg, and $(\lambda_{F_0})_f = -9.1906 \times 10^{-3}$ s/rad using the F_0 formulation, and with $a_f = 41999.999$ km, $e_f = 1.000031$ deg, $i_f = 1.000002$ deg, $\Omega_f = 8.72 \times 10^{-6}$ deg, $\omega_f = 359.999704$ deg, $M_f = 46.114676$ deg, and $(\lambda_{\lambda_0})_f = -1.683 \times 10^{-5}$ s/rad using the λ_0 formulation. Figure 12 shows how the λ_{F_0} and λ_{λ_0} profiles are no longer identical because e_0 is not equal to zero.

Conclusion

The mechanics of trajectory optimization based on the use of the epoch eccentric longitude has been presented as a fundamental formulation from which other formulations, based on various element sets, are mathematically derived. Besides the theoretical insight that this formulation provides into the derivations of the various adjoint differential equations and their underlying assumptions, the analysis further reveals that the adjoint equations need not be integrated during the coasting phases in minimum-fuel problems. Also, the function that defines the transversality condition at the end time in minimum-time problems can effectively be used as a numerical check in accepting a converged trajectory as a truly optimal trajectory because it stays constant during the whole transfer. Further, comparisons between the averaged solutions of this formulation and the current time formulations developed earlier are necessary to evaluate the robustness of the convergence process as well as the computational effort in generating optimized transfers for a variety of transfer geometries.

Appendix: Partial Derivatives of M^{F_0} Matrix with Respect to Orbit Elements

Nonzero Partial Derivatives of M^{F_0} with Respect to h

The partial derivatives $\partial M_{11}/\partial h$, $\partial M_{12}/\partial h$, $\partial M_{21}/\partial h$, $\partial M_{22}/\partial h$, $\partial M_{23}/\partial h$, $\partial M_{31}/\partial h$, $\partial M_{32}/\partial h$, $\partial M_{33}/\partial h$, $\partial M_{43}/\partial h$, and $\partial M_{53}/\partial h$ are identical to Eqs. (A1), (A2), and (A4–A11) of Ref. 17, whereas

$$\frac{\partial M_{61}^0}{\partial h} = n^{-1}a^{-2}\Bigg\{-2\frac{\partial X_1}{\partial h} + 3\frac{\partial \dot{X}_1}{\partial h}t - h\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial k} + \beta\left(h\frac{\partial^2 X_1}{\partial h^2} + k\frac{\partial^2 X_1}{\partial h\partial k}\right)\right]\Bigg\} \tag{A1}$$

$$\frac{\partial M_{62}^0}{\partial h} = n^{-1}a^{-2}\Bigg\{-2\frac{\partial Y_1}{\partial h} + 3\frac{\partial \dot{Y}_1}{\partial h}t - h\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial k} + \beta\left(h\frac{\partial^2 Y_1}{\partial h^2} + k\frac{\partial^2 Y_1}{\partial h\partial k}\right)\right]\Bigg\} \tag{A2}$$

$$\frac{\partial M_{63}^0}{\partial h} = \frac{G^{-1}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right) + hG^{-2}(qY_1 - pX_1)\right] \tag{A3}$$

Nonzero Partial Derivatives of M^{F_0} with Respect to k

The partial derivatives $\partial M_{11}/\partial k$, $\partial M_{12}/\partial k$, $\partial M_{21}/\partial k$, $\partial M_{22}/\partial k$, $\partial M_{23}/\partial k$, $\partial M_{31}/\partial k$, $\partial M_{32}/\partial k$, $\partial M_{33}/\partial k$, $\partial M_{43}/\partial k$, and $\partial M_{53}/\partial k$ are identical to Eqs. (A15), (A16), and (A18–A25) of Ref. 17, whereas

$$\frac{\partial M_{61}^0}{\partial k} = n^{-1}a^{-2}\Bigg\{-2\frac{\partial X_1}{\partial k} + 3\frac{\partial \dot{X}_1}{\partial k}t - k\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial h} + \beta\left(h\frac{\partial^2 X_1}{\partial k\partial h} + k\frac{\partial^2 X_1}{\partial k^2}\right)\right]\Bigg\} \tag{A4}$$

$$\frac{\partial M_{62}^0}{\partial k} = n^{-1}a^{-2}\Bigg\{-2\frac{\partial Y_1}{\partial k} + 3\frac{\partial \dot{Y}_1}{\partial k}t - k\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial h} + \beta\left(h\frac{\partial^2 Y_1}{\partial k\partial h} + k\frac{\partial^2 Y_1}{\partial k^2}\right)\right]\Bigg\} \tag{A5}$$

$$\frac{\partial M_{63}^0}{\partial k} = \frac{G^{-1}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right) + kG^{-2}(qY_1 - pX_1)\right] \tag{A6}$$

Nonzero Partial Derivatives of M^{F_0} with Respect to p

The partial derivatives $\partial M_{23}/\partial p$, $\partial M_{33}/\partial p$, $\partial M_{43}/\partial p$, and $\partial M_{53}/\partial p$ are identical to Eqs. (A29–A32) of Ref. 17, whereas

$$\frac{\partial M_{63}^0}{\partial p} = \frac{-X_1}{na^2G} \tag{A7}$$

Nonzero Partial Derivatives of M^{F_0} with Respect to q

The partial derivatives $\partial M_{23}/\partial q$, $\partial M_{33}/\partial q$, $\partial M_{43}/\partial q$, and $\partial M_{53}/\partial q$ are identical to Eqs. (A34–A37) of Ref. 17, whereas

$$\frac{\partial M_{63}^0}{\partial q} = \frac{Y_1}{na^2G} \tag{A8}$$

The partial derivatives of $\dot{X}_1$ with respect to h and k, namely, $\partial\dot{X}_1/\partial h$ and $\partial\dot{X}_1/\partial k$, are identical to Eqs. (A39) and (A40) of Ref. 17, whereas the partials $\partial\dot{Y}_1/\partial h$ and $\partial\dot{Y}_1/\partial k$ are identical to Eqs. (A41) and (A42) of the same reference and are not repeated here. The second partials of X_1 and Y_1 with respect to h and k, namely, $\partial^2 X_1/\partial h^2$, $\partial^2 X_1/\partial k^2$, $\partial^2 X_1/\partial h\partial k$, and $\partial^2 X_1/\partial k\partial h$, and $\partial^2 Y_1/\partial h^2$, $\partial^2 Y_1/\partial k^2$, $\partial^2 Y_1/\partial h\partial k$, and $\partial^2 Y_1/\partial k\partial h$, are identical to Eqs. (A43–A46) and Eqs. (A47–A50), respectively, of Ref. 17. It can be shown that $\partial^2 X_1/\partial h\partial k = \partial^2 X_1/\partial k\partial h$ and $\partial^2 Y_1/\partial h\partial k = \partial^2 Y_1/\partial k\partial h$. Next, the accessory partials $\partial^2 X_1/\partial a\partial k$, $\partial^2 X_1/\partial a\partial h$, $\partial^2 Y_1/\partial a\partial k$, and $\partial^2 Y_1/\partial a\partial h$ are

$$\frac{\partial^2 X_1}{\partial a\partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} - \frac{3}{2}\frac{nat}{r}\left[(hs_F + kc_F)\frac{hk\beta^3}{(1-\beta)} + \frac{a^2}{r^2}(k - c_F)(s_F - h\beta) - \frac{a}{r}s_F c_F\right] \tag{A9}$$

$$\frac{\partial^2 X_1}{\partial a\partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} - \frac{3}{2}\frac{nat}{r}\left\{(hs_F + kc_F)\left[\beta + \frac{h^2\beta^3}{(1-\beta)}\right] - \frac{a^2}{r^2}(h - s_F)(h\beta - s_F) + \frac{a}{r}c_F^2\right\} \tag{A10}$$

also

$$\frac{\partial^2 Y_1}{\partial a\partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} + \frac{3}{2}\frac{nat}{r}\left\{(hs_F + kc_F)\left[\beta + \frac{k^2\beta^3}{(1-\beta)}\right] + \frac{a^2}{r^2}(k - c_F)(c_F - k\beta) + \frac{a}{r}s_F^2\right\} \tag{A11}$$

$$\frac{\partial^2 Y_1}{\partial a\partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h} + \frac{3}{2}\frac{nat}{r}\left\{(hs_F + kc_F)\frac{hk\beta^3}{(1-\beta)} - \frac{a^2}{r^2}(h - s_F)(k\beta - c_F) - \frac{a}{r}s_F c_F\right\} \tag{A12}$$

Nonzero Partial Derivatives of M^{F_0} with Respect to a

$$\frac{\partial M_{11}}{\partial a} = \frac{4}{n^2a^2}\dot{X}_1 + \frac{2}{n^2a}\frac{\partial \dot{X}_1}{\partial a} \tag{A13}$$

$$\frac{\partial M_{12}}{\partial a} = \frac{4}{n^2a^2}\dot{Y}_1 + \frac{2}{n^2a}\frac{\partial \dot{Y}_1}{\partial a} \tag{A14}$$

$$\frac{\partial M_{21}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial k} + \frac{\partial^2 X_1}{\partial a\partial k} - \frac{h\beta}{na}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A15}$$

$$\frac{\partial M_{22}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial k} + \frac{\partial^2 Y_1}{\partial a\partial k} - \frac{h\beta}{na}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A16}$$

$$\frac{\partial M_{23}}{\partial a} = \frac{k}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A17}$$

$$\frac{\partial M_{31}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial h} + \frac{\partial^2 X_1}{\partial a\partial h} + \frac{k\beta}{na}\dot{X}_1 + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A18}$$

$$\frac{\partial M_{32}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial h} + \frac{\partial^2 Y_1}{\partial a\partial h} + \frac{k\beta}{na}\dot{Y}_1 + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A19}$$

$$\frac{\partial M_{33}}{\partial a} = \frac{-h}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A20}$$

$$\frac{\partial M_{43}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}Y_1 + \frac{\partial Y_1}{\partial a}\right) \tag{A21}$$

$$\frac{\partial M_{53}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}X_1 + \frac{\partial X_1}{\partial a}\right) \tag{A22}$$

$$\frac{\partial M_{61}^0}{\partial a} = -\frac{M_{61}^0}{2a} + n^{-1}a^{-2}\left[-2\frac{\partial X_1}{\partial a} + 3\frac{\partial \dot{X}_1}{\partial a}t + G\left(h\beta\frac{\partial^2 X_1}{\partial a \partial h} + k\beta\frac{\partial^2 X_1}{\partial a \partial k}\right)\right] \tag{A23}$$

$$\frac{\partial M_{62}^0}{\partial a} = -\frac{M_{62}^0}{2a} + n^{-1}a^{-2}\left[-2\frac{\partial Y_1}{\partial a} + 3\frac{\partial \dot{Y}_1}{\partial a}t + G\left(h\beta\frac{\partial^2 Y_1}{\partial a \partial h} + k\beta\frac{\partial^2 Y_1}{\partial a \partial k}\right)\right] \tag{A24}$$

$$\frac{\partial M_{63}^0}{\partial a} = -\frac{M_{63}^0}{2a} + \frac{1}{na^2}\left[\left(q\frac{\partial Y_1}{\partial a} + p\frac{\partial X_1}{\partial a}\right)G^{-1}\right] \tag{A25}$$

with

$$\frac{\partial \dot{X}_1}{\partial a} = -\frac{1}{2}\frac{na}{r}\left[1 - \frac{3na^2t}{r^2}(ks_F - hc_F)\right] \times \left[hk\beta c_F - (1-h^2\beta)s_F\right] + \frac{3}{2}\frac{n^2a^2t}{r^2}\left[hk\beta s_F + (1-h^2\beta)c_F\right] \tag{A26}$$

$$\frac{\partial \dot{Y}_1}{\partial a} = \frac{1}{2}\frac{na}{r}\left[1 - \frac{3na^2t}{r^2}(ks_F - hc_F)\right] \times \left[hk\beta s_F - (1-k^2\beta)c_F\right] + \frac{3}{2}\frac{n^2a^2t}{r^2}\left[hk\beta c_F + (1-k^2\beta)s_F\right] \tag{A27}$$

$$\frac{\partial X_1}{\partial a} = \frac{X_1}{a} - \frac{3}{2}\frac{t}{a}\dot{X}_1 \tag{A28}$$

$$\frac{\partial Y_1}{\partial a} = \frac{Y_1}{a} - \frac{3}{2}\frac{t}{a}\dot{Y}_1 \tag{A29}$$

Nonzero Partial Derivatives of M^{F_0} with Respect to F_0

$$\frac{\partial M_{11}}{\partial F_0} = \frac{\partial M_{11}}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial F_0} = \frac{\partial M_{11}}{\partial \lambda}\frac{r_0}{a_0} = \frac{r_0}{a_0}\frac{2}{n^2 r}\frac{\partial \dot{X}_1}{\partial F} \tag{A30}$$

$$\frac{\partial M_{12}}{\partial F_0} = \frac{r_0}{a_0}\frac{2}{n^2 r}\frac{\partial \dot{Y}_1}{\partial F} \tag{A31}$$

$$\frac{\partial M_{21}}{\partial F_0} = \frac{r_0}{a_0}\frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A32}$$

$$\frac{\partial M_{22}}{\partial F_0} = \frac{r_0}{a_0}\frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A33}$$

$$\frac{\partial M_{23}}{\partial F_0} = \frac{r_0}{a_0}\left[k\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\Big/ narG\right] \tag{A34}$$

$$\frac{\partial M_{31}}{\partial F_0} = -\frac{r_0}{a_0}\frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A35}$$

$$\frac{\partial M_{32}}{\partial F_0} = -\frac{r_0}{a_0}\frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A36}$$

$$\frac{\partial M_{33}}{\partial F_0} = \frac{r_0}{a_0}\left[-h\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\Big/ narG\right] \tag{A37}$$

$$\frac{\partial M_{43}}{\partial F_0} = \frac{r_0}{a_0}\frac{K}{2narG}\frac{\partial Y_1}{\partial F} \tag{A38}$$

$$\frac{\partial M_{53}}{\partial F_0} = \frac{r_0}{a_0}\frac{K}{2narG}\frac{\partial X_1}{\partial F} \tag{A39}$$

$$\frac{\partial M_{61}^0}{\partial F_0} = \frac{r_0}{a_0}n^{-1}a^{-1}r^{-1}\left[-2\frac{\partial X_1}{\partial F} + 3\frac{\partial \dot{X}_1}{\partial F}t + G\left(h\beta\frac{\partial^2 X_1}{\partial F \partial h} + k\beta\frac{\partial^2 X_1}{\partial F \partial k}\right)\right] \tag{A40}$$

$$\frac{\partial M_{62}^0}{\partial F_0} = \frac{r_0}{a_0}n^{-1}a^{-1}r^{-1}\left[-2\frac{\partial Y_1}{\partial F} + 3\frac{\partial \dot{Y}_1}{\partial F}t + G\left(h\beta\frac{\partial^2 Y_1}{\partial F \partial h} + k\beta\frac{\partial^2 Y_1}{\partial F \partial k}\right)\right] \tag{A41}$$

$$\frac{\partial M_{63}^0}{\partial F_0} = \frac{r_0}{a_0}\left[\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\Big/ narG\right] \tag{A42}$$

with $\partial M_{ij}/\partial \lambda_0 = a_0/r_0\, \partial M_{ij}/\partial F_0$. The auxiliary partials are

$$\frac{\partial X_1}{\partial F} = a\left[hk\beta c_F - (1-h^2\beta)s_F\right] \tag{A43}$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta s_F + (1-k^2\beta)c_F\right] \tag{A44}$$

$$\frac{\partial \dot{X}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{X}_1 + \frac{a^2n}{r}\left[-hk\beta s_F - (1-h^2\beta)c_F\right] \tag{A45}$$

$$\frac{\partial \dot{Y}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{Y}_1 + \frac{a^2n}{r}\left[-hk\beta c_F - (1-k^2\beta)s_F\right] \tag{A46}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(hs_F + kc_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(h\beta - s_F)(s_F - h) + \frac{a}{r}c_F^2\right] \tag{A47}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - k) + \frac{a}{r}s_F c_F\right] \tag{A48}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} - \frac{a^2}{r^2}(k\beta - c_F)(s_F - h) + \frac{a}{r}s_F c_F\right] \tag{A49}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[-(hs_F + kc_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(c_F - k\beta)(c_F - k) - \frac{a}{r}s_F^2\right] \tag{A50}$$

Acknowledgment

This work was supported by the U.S. Air Force Space and Missile Systems Center under Contract F-04701-88-C-0089.

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